The Degenerating Behavior of Jacobi's Theta Functions *

Klaus Schiefermayr†

Abstract

In this note, we give the degenerating behavior of Jacobi's theta functions as the modulus k tends to one.

Mathematics Subject Classification: 33E05; 11F27
Keywords: Jacobian elliptic function, Jacobian theta function, Jacobian zeta function

Let k, $0<k<1$, be the modulus of Jacobi's elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$, which are usually defined by a quotient of two theta functions, see [1, Equation (1052.02)] or [3, Equations (2.1.1)–(2.1.3)],

$$\mathrm{sn}(u,k) := \frac{1}{\sqrt{k}}\cdot\frac{\mathrm{H}(u,k)}{\Theta(u,k)},\quad \mathrm{cn}(u,k) := \frac{\sqrt{k'}}{\sqrt{k}}\cdot\frac{\mathrm{H}_1(u,k)}{\Theta(u,k)},\quad \mathrm{dn}(u,k) := \sqrt{k'}\cdot\frac{\Theta_1(u,k)}{\Theta(u,k)}, \tag{1}$$

where $k' := \sqrt{1-k^2}$ is the complementary modulus. The four theta functions of Jacobi are defined by Fourier series, see [1, Equation (1050.01)] or [3, Equations (1.2.11)–(1.2.14)],

$$\begin{aligned}
\Theta(u,k) &:= \theta_4(v,q) := 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nv),\\
\mathrm{H}(u,k) &:= \theta_1(v,q) := 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin((2n+1)v),\\
\mathrm{H}_1(u,k) &:= \theta_2(v,q) := 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos((2n+1)v),\\
\Theta_1(u,k) &:= \theta_3(v,q) := 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nv),
\end{aligned} \tag{2}$$

where $v=\pi u/(2K)$ and $q=q(k):=\exp(-\pi K'/K)$ is the nome of Jacobi's theta functions. Here, we have used the old notation of the theta functions, which goes back to Jacobi, because the relation between Jacobi's elliptic and theta functions is better visible, see formulae (1).

Furthermore, $K=K(k)$ is the complete elliptic integral of the first kind and $K' = K'(k) := K(k')$. Finally, we will need Jacobi's zeta function, denoted by $\mathrm{zn}(u,k)$, which is usually defined by, see [1, Equation (1053.01)] or [3, Equation (3.6.1)],

$$\mathrm{zn}(u,k) := \frac{\partial}{\partial u}\log\Theta(u,k). \tag{3}$$

*published in: *Integral Transforms and Special Functions* **26** (2015), 226–228.
†University of Applied Sciences Upper Austria, Campus Wels, School of Engineering, Austria, KLAUS.SCHIEFERMAYR@FH-WELS.AT

Here we follow the notation of Carlson and Todd [2], in other references, like [1] and [3], Jacobi's zeta function is denoted by $\mathrm{Z}(u,k)$. As a standard reference for Jacobian elliptic and theta functions, we recommend [1], [3], and [6].

If the modulus k tends to zero then it follows immediately from the definitions that $\mathrm{zn}(u,k)$, $\mathrm{H}(u,k)$, and $\mathrm{H}_1(u,k)$ tend to zero, whereas $\Theta(u,k)$ and $\Theta_1(u,k)$ tend to one. In this note, we give the behavior of Jacobi's zeta function $\mathrm{zn}(u,k)$ and of Jacobi's four theta functions $\Theta(u,k)$, $\mathrm{H}(u,k)$, $\mathrm{H}_1(u,k)$, $\Theta_1(u,k)$ for $k\to 1$. Although the limiting values of Jacobi's elliptic functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ as $k\to 1$ are given in many textbooks like [4], corresponding results for Jacobi's theta functions seem to be lacking.

Theorem 1. *Given $u\in\mathbb{R}$, for $k\to 1$, we have*

$$\begin{aligned}
\mathrm{zn}(u,k) &\sim \tanh(u)\\
\frac{\Theta(u,k)}{\Theta(0,k)} &\sim \cosh(u),\\
\frac{\mathrm{H}(u,k)}{\Theta(0,k)} &\sim \sinh(u),\\
\frac{\mathrm{H}_1(u,k)}{\Theta_1(0,k)} &\sim \frac{\Theta_1(u,k)}{\Theta_1(0,k)} \sim 1,
\end{aligned}\tag{4}$$

where $f(u,k)\sim g(u,k)$ means that $\lim_{k\to 1} f(u,k)/g(u,k)=1$.

The proof of Theorem 1 is mainly based on the following formula for $\Theta(u,k)$; see [6, Section 22.74]. For a similar formula for $\mathrm{H}(u,k)$, see the remark after Lemma 2 in [5].

Lemma 1. *For $v,w\in\mathbb{R}$, we have*

$$\log\frac{\Theta(v-w)}{\Theta(v+w)} = \int_0^v \frac{2k^2\,\mathrm{sn}^2(\xi,k)\,\mathrm{sn}(w,k)\,\mathrm{cn}(w,k)\,\mathrm{dn}(w,k)}{1-k^2\,\mathrm{sn}^2(\xi,k)\,\mathrm{sn}^2(w,k)}\,\mathrm{d}\xi - 2\,v\,\mathrm{zn}(w,k). \tag{5}$$

Furthermore, we also need the behavior of $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, and $\mathrm{dn}(u,k)$ for $k\to 1$, see [1, Equation (127.02)] or [3, Equation (2.6.17)] or [4, Table 22.5.4].

Lemma 2. *Given $u\in\mathbb{R}$, for $k\to 1$, we have*

$$\begin{aligned}
\mathrm{sn}(u,k) &\sim \tanh(u),\\
\mathrm{cn}(u,k) &\sim \mathrm{dn}(u,k) \sim \frac{1}{\cosh(u)}.
\end{aligned}\tag{6}$$

Lemma 3. *For $v,w\in\mathbb{R}$, we have*

$$\frac{2\tanh(w)}{\cosh^2(w)}\int_0^v \frac{\tanh^2(\xi)}{1-\tanh^2(w)\tanh^2(\xi)}\,\mathrm{d}\xi = \log\frac{\cosh(v-w)}{\cosh(v+w)} + 2\,v\tanh(w). \tag{7}$$

Proof. Formula (7) follows immediately by differentiating both sides with respect to v and using the addition formulae for $\tanh(v\pm w)$. □

Proof of Theorem 1. Carlson and Todd [2, Theorem 4] proved the relation

$$\mathrm{zn}(u,k) = \mathrm{sn}(u,k) - u/K + r \quad \text{with } |r|\le {k'}^2 u(1-u/K),\ 0\le u\le K.$$

Thus, for fixed $u\in\mathbb{R}$, we have $\mathrm{zn}(u,k)\sim\mathrm{sn}(u,k)$ as $k\to 1$ (note that $K\to\infty$) and the first relation of (4) follows by (6).

Concerning the second relation of (4), let $v, w \in \mathbb{R}$ be fixed. Then, as $k \to 1$, we have

$$\log\frac{\Theta(v-w)}{\Theta(v+w)} = \int_0^v \frac{2k^2\,\mathrm{sn}^2(\xi,k)\,\mathrm{sn}(w,k)\,\mathrm{cn}(w,k)\,\mathrm{dn}(w,k)}{1-k^2\,\mathrm{sn}^2(\xi,k)\,\mathrm{sn}^2(w,k)}\,\mathrm{d}\xi - 2\,v\,\mathrm{zn}(w,k) \qquad \text{by (5)}$$
$$\sim \frac{2\tanh(w)}{\cosh^2(w)}\int_0^v \frac{\tanh^2(\xi)}{1-\tanh^2(w)\tanh^2(\xi)}\,\mathrm{d}\xi - 2\,v\tanh(w) \qquad \text{by (6)}$$
$$= \log\frac{\cosh(v-w)}{\cosh(v+w)} \qquad \text{by Lemma 3.}$$

Setting $v = w = u/2$, we get

$$\frac{\Theta(0,k)}{\Theta(u,k)} \sim \frac{1}{\cosh(u)}.$$

Using (1), (6), and the identities [1, Equation (1051.01)]

$$\Theta(0,k) = \sqrt{k'}\,\Theta_1(0,k) = \sqrt{2k'K/\pi},$$

we get

$$\begin{aligned}
\frac{\mathrm{H}(u,k)}{\Theta(0,k)} &= \sqrt{k}\,\mathrm{sn}(u,k)\frac{\Theta(u,k)}{\Theta(0,k)} \sim \tanh(u)\cosh(u) = \sinh(u),\\
\frac{\mathrm{H}_1(u,k)}{\Theta_1(0,k)} &= \mathrm{cn}(u,k)\frac{\Theta(u,k)}{\Theta(0,k)} \sim \frac{1}{\cosh(u)}\cdot\cosh(u) = 1,\\
\frac{\Theta_1(u,k)}{\Theta_1(0,k)} &= \mathrm{dn}(u,k)\frac{\Theta(u,k)}{\Theta(0,k)} \sim \frac{1}{\cosh(u)}\cdot\cosh(u) = 1.
\end{aligned}$$

□

References

[1] P.F. Byrd and M.D. Friedman, *Handbook of elliptic integrals for engineers and scientists*, Springer, 1971.

[2] B.C. Carlson and J. Todd, *The degenerating behavior of elliptic functions*, SIAM J. Numer. Anal. **20** (1983), 1120–1129.

[3] D.F. Lawden, *Elliptic functions and applications*, Springer, 1989.

[4] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, and C.W. Clark (eds.), *NIST handbook of mathematical functions*, U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, 2010.

[5] K. Schiefermayr, *Some new properties of Jacobi's theta functions*, J. Comput. Appl. Math. **178** (2005), 419–424.

[6] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge University Press, 1962.