Inverse Polynomial Images and Inequalities for Polynomials Minimal with respect to the Maximum Norm

HABILITATION THESIS

submitted to the
Faculty of Engineering and Natural Sciences
Johannes Kepler University Linz

by

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Wels, August 2010

To my children FLORIAN, JULIA and SOPHIE

Preface

This habilitation thesis is a collection of 8 articles on approximation theory and complex function theory, which have all been published in international mathematical journals. One part of the thesis is concerned with the description of polynomials, whose inverse image consists of two Jordan arcs, with the help of elliptic functions, the other part deals with some inequalities for the norm of polynomials, which are minimal with respect to the maximum norm. A short summary, which emphasizes the connection between the 8 publications, should help readers to better orientate themselves.

I would like to express my special thanks to two persons who had a great influence on my scientific career, Prof. Dr. Franz Peherstorfer, who recently passed away, and Prof. Dr. Peter Weiß. I am also grateful to my parents for giving me the opportunity to study and for their constant support.

To my wife Sonja, I would like to express my deepest gratitude for her loving support during the long process of writing this thesis.

I would like to dedicate this thesis to our loving children Florian, Julia and Sophie.

Wels, August 2010 Klaus Schiefermayr

Summary of the Habilitation Thesis

This is a summary of the following 8 articles, which have been published in international journals:

[H1] Klaus Schiefermayr, A lower bound for the norm of the minimal residual polynomial, *Constructive Approximation* (2010), in press.

[H2] Klaus Schiefermayr, Estimates for the asymptotic convergence factor of two intervals, *Journal of Computational and Applied Mathematics* (2010), in press.

[H3] Klaus Schiefermayr, Inverse polynomial images consisting of an interval and an arc, *Computational Methods and Function Theory* **9** (2009), 407–420.

[H4] Klaus Schiefermayr, A lower bound for the minimum deviation of the Chebyshev polynomial on a compact real set, *East Journal on Approximations* **14** (2008), 223–233.

[H5] Klaus Schiefermayr, An upper bound for the logarithmic capacity of two intervals, *Complex Variables and Elliptic Equations* **53** (2008), 65–75.

[H6] Klaus Schiefermayr, Inverse polynomial images which consists of two Jordan arcs – an algebraic solution, *Journal of Approximation Theory* **148** (2007), 148–157.

[H7] Klaus Schiefermayr, Some new properties of Jacobi's theta functions, *Journal of Computational and Applied Mathematics* **178** (2005), 419–424.

[H8] Franz Peherstorfer and Klaus Schiefermayr, Description of inverse polynomial images which consist of two Jordan arcs with the help of Jacobi's elliptic functions, *Computational Methods and Function Theory* **4** (2004), 355–390.

Note that these 8 papers are cited as [H1], [H2], ..., [H8]. The citations [1], [2], etc. refer to the bibliography at the end of this summary.

1 Inverse Polynomial Images

For $n \in \mathbb{N}$, let $\mathbb{P}_n$ denote the set of all polynomials of degree n with complex coefficients. For a polynomial $P_n \in \mathbb{P}_n$, let $P_n^{-1}(\mathbb{R})$ be the inverse image of $\mathbb{R}$ under the polynomial mapping P_n, i.e.,

$$P_n^{-1}(\mathbb{R}) = \bigl\{z \in \mathbb{C} : P_n(z) \in \mathbb{R}\bigr\}. \tag{1}$$

It was already known to Gauß, since it is central in his proof of the "Fundamental Theorem of Algebra", that $P_n^{-1}(\mathbb{R})$ consists of n analytic Jordan arcs moving from ∞ to ∞, which cross each other only at those zeros of the derivative P_n' which are real. Note that two Jordan arcs from $P_n^{-1}(\mathbb{R})$ cross each other at most once. As usual, a set $\bigl\{\gamma(t) \in \overline{\mathbb{C}} : t \in [0,1]\bigr\}$ is called a *Jordan arc* if $\gamma : [0,1] \to \overline{\mathbb{C}}$ is continuous and $\gamma : [0,1) \to \overline{\mathbb{C}}$ is injective. If, in addition, $\gamma'(t)$ exists and is continuous in $[0,1]$ then it is called an *analytic Jordan arc.*

Subsequently, the inverse image of $[-1,1]$ under the polynomial mapping P_n, i.e.,

$$P_n^{-1}([-1,1]) = \bigl\{z \in \mathbb{C} : P_n(z) \in [-1,1]\bigr\}, \tag{2}$$

is obtained by cutting off the n arcs of $P_n^{-1}(\mathbb{R})$. Clearly, $P_n^{-1}([-1,1])$ consists of n analytic Jordan arcs. However, the arcs can be chosen such that $P_n^{-1}([-1,1])$ consists of n Jordan arcs (not necessarily analytic), on which P_n is strictly monotone increasing from -1 to $+1$, see [23]. If there is a point $z_1 \in \mathbb{C}$, for which $P_n(z_1) \in \{-1,1\}$ and $P_n'(z_1) = 0$, then two Jordan arcs can be combined into one Jordan arc. This combination of arcs can be seen very clearly with the help of the inverse image of the classical Chebyshev polynomial of the first kind, $T_n(z) := \cos(n \arccos z)$. In this case,

$$T_n^{-1}([-1,1]) = [-1,1], \tag{3}$$

i.e., $T_n^{-1}([-1,1])$ is *one* Jordan arc, since there are $n-1$ points z_j with the two properties $T_n(z_j) \in \{-1,1\}$ and $T_n'(z_j) = 0$. In the left plot of Fig. 1, we have plotted the graph of $T_n(z)$, $n = 9$, for $z \in [-1,1]$, in the right plot of Fig. 1, the sets $T_n^{-1}(\mathbb{R})$ (dotted and thick line) and $T_n^{-1}([-1,1]) = [-1,1]$ (thick line) are shown.

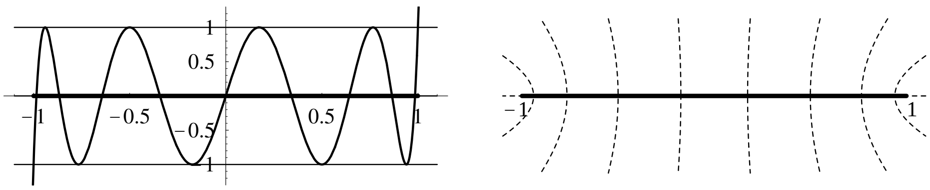

Figure 1: The graph of the classical Chebyshev polynomial T_n for $n = 9$ and its inverse images $T_n^{-1}(\mathbb{R})$ and $T_n^{-1}([-1,1]) = [-1,1]$

Note that the classical Chebyshev polynomial T_n is, up to a linear transformation, the only polynomial whose inverse image of $[-1,1]$ is *one* Jordan arc, see [H8, Remark 4]. Moreover, by (3), for any polynomial $P_n \in \mathbb{P}_n$,

$$(T_m \circ P_n)^{-1}([-1,1]) = P_n^{-1}([-1,1]). \tag{4}$$

In [H8, Theorem 3], we gave a necessary and sufficient condition in form of a polynomial equation such that $\mathcal{T}_n^{-1}([-1,1])$ consists of 2 Jordan arcs, compare also [19]. The condition and the proof can be easily extended to the general case of ℓ arcs, compare also [23, Remark after Corollary 2.2].

Lemma 1. *For any polynomial $\mathcal{T}_n(z) = c_n z^n + \ldots \in \mathbb{P}_n$, $c_n \in \mathbb{C} \setminus \{0\}$, there exists a unique $\ell \in \{1,2,\ldots,n\}$, a unique monic polynomial $\mathcal{H}_{2\ell}(z) = z^{2\ell} + \ldots \in \mathbb{P}_{2\ell}$ with pairwise distinct zeros $a_1, a_2, \ldots, a_{2\ell}$, i.e.,*

$$\mathcal{H}_{2\ell}(z) = \prod_{j=1}^{2\ell} (z - a_j), \tag{5}$$

and a unique polynomial $\mathcal{U}_{n-\ell}(z) = c_n z^{n-\ell} + \ldots \in \mathbb{P}_{n-\ell}$ with the same leading coefficient c_n such that the polynomial equation

$$\mathcal{T}_n^2(z) - 1 = \mathcal{H}_{2\ell}(z)\,\mathcal{U}_{n-\ell}^2(z) \tag{6}$$

holds. Note that the points $a_1, a_2, \ldots, a_{2\ell}$ are exactly those zeros of $\mathcal{T}_n^2 - 1$ which have odd multiplicity.

Theorem 1 (Peherstorfer and Schiefermayr [H8]). *Let $\mathcal{T}_n \in \mathbb{P}_n$. Then $\mathcal{T}_n^{-1}([-1,1])$ consists of ℓ (but not less than ℓ) Jordan arcs with endpoints $a_1, a_2, \ldots, a_{2\ell}$ if and only if $\mathcal{T}_n^2 - 1$ has exactly 2ℓ pairwise distinct zeros $a_1, a_2, \ldots, a_{2\ell}$, $1 \le \ell \le n$, of odd multiplicity, i.e., if and only if $\mathcal{T}_n$ satisfies a polynomial equation of the form* (6) *with $\mathcal{H}_{2\ell}$ given in* (5)*.*

In [H8] and [H7], we considered polynomials $\mathcal{T}_n$, whose inverse image $\mathcal{T}_n^{-1}([-1,1])$ consists of *two* Jordan arcs. Starting point was the polynomial equation (6) (with $\ell = 2$). With the help of Jacobi's elliptic and theta functions, we gave a necessary and sufficient condition for four distinct points $a_1, a_2, a_3, a_4 \in \mathbb{C}$ to be the endpoints of of the inverse image (consisting of two Jordan arcs) of a polynomial of degree n. The three cases, when (i) all four points are real, (ii) two points are real and two points are complex conjugate, and (iii) two pairs of complex conjugate points, are considered in detail. The results are summarized in Section 5.

Of special interest are inverse polynomial images which consists of a subset of the real line. In [23], Peherstorfer gave a necessary and sufficient condition for the polynomial P_n such that $P_n^{-1}([-1,1])$ is real. Fig. 2 shows the graph of a polynomial P_n with degree $n = 9$, for which $P_n^{-1}([-1,1])$ consists of $\ell = 4$ intervals.

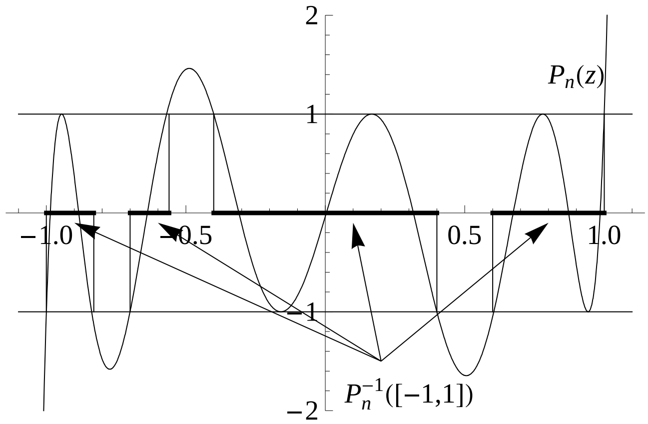

Figure 2: Polynomial P_n of degree $n = 9$, for which the inverse polynomial image consists of four real intervals

In the following, let $S \subset \mathbb{C}$ be a compact infinite set in the complex plane and let the norm $\|\cdot\|_S$ associated with S be defined by

$$\|P_n\|_S := \max_{z \in S} |P_n(z)|,$$

usually called the maximum norm (or supremum norm) on S. In the following two sections, we consider two classical approximation problems with respect to the maximum norm.

2 The Norm of Minimal Residual Polynomials

Consider the following approximation problem: Let $S \subset \mathbb{C}$ be a compact infinite set in the complex plane and $0 \notin S$. For fixed $n \in \mathbb{N}$, find that polynomial $R_m \in \mathbb{P}_m$, $m \le n$, with $R_m(0) = 1$, for which the maximum norm on S is minimal, i.e.,

$$L_n^r(S,0) := \|R_m\|_S = \min\bigl\{\|P_m\|_S : P_m \in \mathbb{P}_m, m \le n, P_m(0) = 1\bigr\}. \tag{7}$$

The optimal polynomial $R_m \in \mathbb{P}_m$ is unique and usually called the *minimal residual polynomial* for the degree n on S and the quantity $L_n^r(S,0)$ is called the minimum deviation of R_m on S. Note that we say *for* the degree n but not *of* degree n since it turned out that the minimal residual polynomial for the degree n is a polynomial of degree $m=n$ or $m=n-1$ if S is a real set. Indeed, we proved the following [H1, Lemma 2 and Corollary 2]:

Theorem 2 (Schiefermayr [H1])**.** *Let $n \in \mathbb{N}$, let $S \subset \mathbb{R}$ be compact with at least $n+1$ points and $0 \notin S$, let $R_m \in \mathbb{P}_m$, $m \leq n$, be the minimal residual polynomial for the degree n on S with minimum deviation $L_n^r(S,0)$, and define $P_m := R_m/L_n^r(S,0)$. Then $P_m^{-1}([-1,1])$ is the union of ℓ finite disjoint real intervals, where $1 \leq \ell \leq n$ and $S \subseteq A \subset \mathbb{R}$. Moreover, the minimal residual polynomial R_m is a polynomial of degree $m=n$ or of degree $m=n-1$.*

It is well known, see [15] or [7], that the limit

$$\kappa(S,0) := \lim_{n\to\infty} \sqrt[n]{L_n^r(S,0)} \tag{8}$$

exists, where $\kappa(S,0)$ is usually called the *estimated asymptotic convergence factor*. This factor $\kappa(S,0)$ has a very nice representation in terms of the corresponding Green's function for the complement of the set S, see [H1, Section 3].

The approximation problem (7) and the convergence factor (8) arise for instance in the context of solving large linear systems $Ax=b$ by Krylov subspace methods, where the spectrum of the matrix A is approximated by the set S, see, e.g., [7], [11], and [15].

For $S \subset \mathbb{C}$, the lower bound

$$L_n^r(S,0) \geq \kappa(S,0)^n$$

is well known and widely referenced, see [15] or [7], and can be proven with the help of the Bernstein-Walsh lemma, see [24, Section 5.5]. In [H1], we obtained a sharper lower bound for the union of a finite number of real intervals:

Theorem 3 (Schiefermayr [H1])**.** *Let $n \in \mathbb{N}$, and let S be the union of a finite number of real intervals with $0 \notin S$. Let $L_n^r(S,0)$ and $\kappa(S,0)$ as in (7) and (8), respectively. Then the inequality*

$$L_n^r(S,0) \geq \frac{2\kappa(S,0)^n}{1+\kappa(S,0)^{2n}} \tag{9}$$

holds. Equality is attained in (9) if and only if there exists a polynomial $P_n \in \mathbb{P}_n$ of degree n such that $S = P_n^{-1}([-1,1])$. If, in addition to $S = P_n^{-1}([-1,1])$, the point zero lies in the convex hull of S, then

$$L_n^r(S,0) = L_{n+1}^r(S,0).$$

Moreover, from the proof of Theorem 3, we obtained a refinement for the Bernstein-Walsh lemma for the case of several real intervals and real arguments [H1, Corollary 3]:

Theorem 4 (Schiefermayr [H1])**.** *Let K be the union of a finite number of real intervals. Then for any polynomial Q_n of degree n,*

$$\frac{|Q_n(x)|}{\|Q_n\|_K} \leq \frac{1}{2}\big(\exp(n\cdot g(x;K^{\mathrm{c}})) + \exp(-n\cdot g(x;K^{\mathrm{c}}))\big) \qquad (x \in \mathbb{R}\setminus K),$$

where $g(x;K^{\mathrm{c}})$ denotes the Green's function of K^{c}. Equality is attained if and only if Q_n is such that $Q_n^{-1}([-1,1]) = K$.

Next, let us turn to the case of two real intervals, i.e.,

$$E := [a_1, a_2] \cup [a_3, a_4], \qquad a_1 < a_2 < a_3 < a_4, \tag{10}$$

where $0 \in \mathbb{R} \setminus E$. It is convenient to use the linear transformation

$$\ell(x) := \frac{2x - a_1 - a_4}{a_4 - a_1},$$

which maps the set E onto the normed set

$$\hat{E} := [-1, \alpha] \cup [\beta, 1], \tag{11}$$

where $\alpha := \ell(a_2)$ and $\beta := \ell(a_3)$. For the norm of the corresponding minimal residual polynomial, we have

$$L_n^r(E, 0) = L_n^r(\hat{E}, \xi) \tag{12}$$

and thus

$$\kappa(E, 0) = \kappa(\hat{E}, \xi), \tag{13}$$

where $\xi := \ell(0)$. The asymptotic convergence factor $\kappa(\hat{E}, \xi)$ can be expressed with the help of Jacobi's elliptic and theta functions (based on the above mentioned representation via the Green function), see [H2, Theorem 1], but this formula is very involved. Hence, in [H2, Theorem 3], we obtained the following upper and lower bound in terms of *elementary functions* of α, β and ξ:

Theorem 5 (Schiefermayr [H2]). *Let $\hat{E} := [-1, \alpha] \cup [\beta, 1]$, $-1 < \alpha < \beta < 1$ and let $\xi \in \mathbb{R} \setminus \hat{E}$. Then, for the convergence factor $\kappa(\hat{E}, \xi)$, the inequalities*

$$\frac{A_2}{A_1} \cdot B \le \kappa(\hat{E}, \xi) \le \frac{A_1}{A_2} \cdot B \tag{14}$$

hold, where

$$\begin{aligned} A_1 &:= \sqrt[4]{(1-\alpha)(1+\beta)} + \sqrt[4]{(1+\alpha)(1-\beta)}, \\ A_2 &:= \sqrt[4]{8}\sqrt[4]{\sqrt{(1-\alpha)(1+\beta)} + \sqrt{(1+\alpha)(1-\beta)}}\,\sqrt[16]{(1-\alpha^2)(1-\beta^2)}, \end{aligned} \tag{15}$$

and B is given in the following:

(i) *For $\alpha < \xi < \beta$,*

$$B := \frac{\sqrt[4]{(1+\alpha)(1-\beta)} + \sqrt{1-\xi} - \sqrt{(\xi-\alpha)(\beta-\xi)}}{\sqrt[4]{(1+\alpha)(1-\beta)} + \sqrt{1-\xi} + \sqrt{(\xi-\alpha)(\beta-\xi)}}. \tag{16}$$

(ii) *For $\xi \in \mathbb{R} \setminus [-1, 1]$,*

$$\begin{aligned} B := &\frac{(2\xi - \xi\alpha + \xi\beta - \alpha - \beta)\sqrt[4]{\frac{(1+\alpha)(1-\beta)}{(1-\alpha)(1+\beta)}} + 2\sqrt{(\xi-\alpha)(\xi-\beta)} - (\beta-\alpha)\sqrt{\xi^2-1}}{(2\xi - \xi\alpha + \xi\beta - \alpha - \beta)\sqrt[4]{\frac{(1+\alpha)(1-\beta)}{(1-\alpha)(1+\beta)}} + 2\sqrt{(\xi-\alpha)(\xi-\beta)} + (\beta-\alpha)\sqrt{\xi^2-1}} \\ &\times \frac{\left|\sqrt{(1+\xi)(\xi-\alpha)} - \sqrt{(\xi-1)(\xi-\beta)}\right|}{\sqrt{(1+\xi)(\xi-\alpha)} + \sqrt{(\xi-1)(\xi-\beta)}} \end{aligned} \tag{17}$$

The estimates for $\kappa(\hat{E},\xi)$ given in Theorem 5 are very precise, see the Remark after Theorem 3 in [H2]. In order to obtain the estimates of Theorem 5, we derived several inequalities and monotonicity properties of Jacobi's elliptic and theta functions, see [H2, Section 6].

Moreover, the following problem is solved in [H2, Section 4]: given the length of the two intervals, say ℓ_1 and ℓ_2, and given the length of the gap between the two intervals, say ℓ_3, for which set of two intervals $E=[a_1,a_2]\cup[a_3,a_4]$, $a_1<a_2<0<a_3<a_4$, with $a_2-a_1=\ell_1$, $a_4-a_3=\ell_2$ and $a_3-a_2=\ell_3$, the convergence factor $\kappa(E,0)$ is minimal?

3 The Norm of Chebyshev Polynomials

Let $\hat{\mathbb{P}}_n$ denote the set of all monic polynomials (polynomials with leading coefficient 1) of degree n with complex coefficients. Consider the following approximation problem: Let $S\subset\mathbb{C}$ be a compact infinite set in the complex plane. Find that polynomial $M_n\in\hat{\mathbb{P}}_n$, for which the maximum norm on S is minimal, i.e.,

$$L_n(S):=\|M_n\|_S=\min\bigl\{\|P_n\|_S : P_n\in\hat{\mathbb{P}}_n\bigr\}. \tag{18}$$

The term $L_n(S)$ is usually called the *minimum deviation* of degree n on S. The polynomial $M_n\in\hat{\mathbb{P}}_n$, for which the minimum is attained, is called the *Chebyshev polynomial* (or minimal polynomial with respect to the maximum norm) of degree n on S. It is well known that the limit

$$\operatorname{cap} S:=\lim_{n\to\infty}\sqrt[n]{L_n(S)} \tag{19}$$

exists and the quantity $\operatorname{cap} S$ is called the *Chebyshev constant* of S. The term $\operatorname{cap} S$ can be given in a completely different way by

$$\operatorname{cap} S=\lim_{n\to\infty}\sup_{z_i,z_j\in S}\Bigl(\prod_{1\le i<j\le n}|z_i-z_j|\Bigr)^{\frac{2}{n(n-1)}}$$

and, in this connection, is called the *transfinite diameter* of S. Finally, $\operatorname{cap} S$ can be defined via the equilibrium measure of S, see [24, Section 5], and is in this context called the *logarithmic capacity* of S. Since in the literature the term logarithmic capacity is used most of all, we remain at this notation. Concerning the properties of the logarithmic capacity and the connection to potential theory, we refer to [14] and [24].

For any given polynomial $P_n(z)=c_nz^n+\ldots\in\mathbb{P}_n$, $c_n\in\mathbb{C}\setminus\{0\}$, the monic polynomial

$$\hat{P}_{\ell n}(z):=\frac{2}{(2c_n)^\ell}\,T_\ell(P_n(z))=z^{\ell n}+\ldots\in\hat{\mathbb{P}}_{\ell n}$$

is the Chebyshev polynomial of degree ℓn, $\ell\in\mathbb{N}$, on the inverse image $S=P_n^{-1}([-1,1])$, see [13], [21], or [18]. From this fact, the following remarkable result follows immediately [H4, Theorem 1]:

Theorem 6 (Schiefermayr [H4])**.** *Let $P_n\in\mathbb{P}_n$ and $S:=P_n^{-1}([-1,1])$. Then*

$$L_n(S)=2\,(\operatorname{cap} S)^n. \tag{20}$$

The inequality

$$L_n(S)\ge(\operatorname{cap} S)^n \tag{21}$$

may be found in many textbooks and papers like [25, Appendix B] and goes back to Szegö [28] and Fekete [9]. In [H4], we obtained the following refinement of (21) for the real case:

Theorem 7 (Schiefermayr [H4])**.** *Let $S \subset \mathbb{R}$ be compact and infinite, then, for each $n \in \mathbb{N}$,*

$$L_n(S) \geq 2\,(\operatorname{cap} S)^n, \tag{22}$$

where equality is attained if there exists a polynomial $P_n \in \mathbb{P}_n$ such that $S = P_n^{-1}([-1,1])$.

In addition, some analogous results concerning compact infinite sets on the unit circle, lying symmetrically with respect to the real line, are given in [H4].

Let us turn to the case of two real intervals $\hat{E} := [-1,\alpha] \cup [\beta,1]$, $-1 < \alpha \leq \beta < 1$. Based on an representation of the logarithmic capacity of $\hat{E}$ in terms of Jacobi's elliptic and theta functions due to Achieser [2], we derived [H2, Theorem 5] the following very simple lower bound for the logarithmic capacity of $\hat{E}$:

Theorem 8 (Schiefermayr [H2])**.** *Let $\hat{E} := [-1,\alpha] \cup [\beta,1]$, $-1 < \alpha \leq \beta < 1$, then*

$$\operatorname{cap} \hat{E} \geq \frac{1}{2}\left(\frac{\sqrt[4]{1-\alpha^2} + \sqrt[4]{1-\beta^2}}{\sqrt[4]{(1-\alpha)(1+\beta)} + \sqrt[4]{(1+\alpha)(1-\beta)}}\right)^4, \tag{23}$$

where equality is attained if $\alpha = \beta$ or if $\alpha \to -1$ (β fixed) or if $\beta \to 1$ (α fixed).

In [27], Solynin gave an excellent lower bound for the logarithmic capacity of the union of several intervals, see also [H4] and [H5] for a discussion of this result. Although we could not achieve the goodness of Solynin's bound in the two interval case, we found it useful to give this very simple lower bound (23). In the recent paper [8], Dubinin and Karp even improved Solynin's lower bound and, in addition, based on a result of Haliste [12], they gave an upper bound for the logarithmic capacity of several intervals. For the two intervals case, the result reads as follows:

Theorem 9 (Dubinin & Karp [8])**.** *Let $\hat{E} := [-1,\alpha] \cup [\beta,1]$, $-1 < \alpha < \beta < 1$, then*

$$\operatorname{cap} \hat{E} \leq \tfrac{1}{4}\Big(\sqrt{(1+\alpha)(1+\beta)} + \sqrt{(1-\alpha)(1-\beta)}\Big), \tag{24}$$

where equality is attained if $\alpha = \beta$ or if $\alpha = -\beta$.

In [H5], another (more complicated) upper bound in terms of elementary functions for $\operatorname{cap} \hat{E}$ is derived. Numerical computations indicated that this upper bound is better than (24) if, for fixed α, the endpoint β is near 1.

4 Inverse Polynomial Images which Consists of Two Jordan Arcs

4.1 The General Characterization via Jacobi's Elliptic and Theta Functions

Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$ be four pairwise distinct points in the complex plane, let

$$\mathcal{H}(z) := (z-a_1)(z-a_2)(z-a_3)(z-a_4), \tag{25}$$

and suppose that $\mathcal{T}_n(z) = c_n z^n + \ldots \in \mathbb{P}_n$ satisfies a polynomial equation of the form

$$\mathcal{T}_n^2(z) - \mathcal{H}(z)\,\mathcal{U}_{n-2}^2(z) = 1, \tag{26}$$

where $\mathcal{U}_{n-2}(z)=c_n z^{n-2}+\ldots\in\mathbb{P}_{n-2}$. Then, as mentioned above (see Lemma 1 and Theorem 1), the inverse image $\mathcal{T}_n^{-1}([-1,1])$ consists of two Jordan arcs with endpoints a_1,a_2,a_3,a_4.

It turned out that the point z^* defined by the equation

$$\mathcal{T}_n'(z)=n(z-z^*)\,\mathcal{U}_{n-2}(z) \tag{27}$$

plays an important part in what follows. With the help of z^*, $\mathcal{T}_n^{-1}([-1,1])$ can be given by the integral [21]

$$\mathcal{T}_n(z)=\pm\cosh\Bigl(n\int_{a_1}^{z}\frac{w-z^*}{\sqrt{\mathcal{H}(w)}}\,\mathrm{d}w\Bigr). \tag{28}$$

Concerning the two Jordan arcs, we proved the following [H8, Theorem 3]: (i) the two Jordan arcs have at most one common point (that is the point z^*) and (ii) the two Jordan arcs are disjoint if and only if $z^*\notin\mathcal{T}_n^{-1}([-1,1])$.

The next result [H3, Theorem 1] gives the number of extremal points of $\mathcal{T}_n$ on $\mathcal{T}_n^{-1}([-1,1])$. As usual, a point $z_0\in\mathcal{T}_n^{-1}([-1,1])$ is called an extremal point of $\mathcal{T}_n$ on $\mathcal{T}_n^{-1}([-1,1])$ if $|\mathcal{T}_n(z_0)|=\max_{z\in\mathcal{T}_n^{-1}([-1,1])}|\mathcal{T}_n(z)|$, i.e., if $\mathcal{T}_n(z_0)\in\{-1,1\}$.

Theorem 10 (Schiefermayr [H3]). *Suppose that the polynomial equation* (26) *holds and let z^* be defined by* (27). *If $\mathcal{T}_n(z^*)\neq\pm1$ $\bigl[\mathcal{T}_n(z^*)=\pm1\bigr]$ then $\mathcal{T}_n$ has $n+2$ $\bigl[n+1\bigr]$ extremal points on $\mathcal{T}_n^{-1}([-1,1])$.*

Let us introduce the following notation. We call four pairwise distinct points $a_1,a_2,a_3,a_4\in\mathbb{C}$ with the property

$$|a_4-a_1|\,|a_3-a_2|\leq|a_4-a_2|\,|a_3-a_1| \tag{29}$$

a $\mathbf{T}_n$-tuple if there exists a polynomial $\mathcal{T}_n\in\mathbb{P}_n$, called the associated polynomial, such that $\mathcal{T}_n^{-1}([-1,1])$ consists of two Jordan arcs with endpoints a_1,a_2,a_3,a_4, or, in other words (by Theorem 1), if there exist polynomials $\mathcal{T}_n\in\mathbb{P}_n$ and $\mathcal{U}_{n-2}\in\mathbb{P}_{n-2}$ such that a polynomial equation of the form (26) holds, where $\mathcal{H}$ is defined in (25). Note that inequality (29) is no loss of generality since for each tuple (a_1,a_2,a_3,a_4), which does not satisfy (29), one only has to exchange a_1 and a_2.

For the characterization of $\mathbf{T}_n$-tuples and the associated polynomial $\mathcal{T}_n$, we used Jacobi's elliptic and theta functions, hence let us recall some notations. Let $k\in D_k$, D_k defined in (31) below, be the modulus of Jacobi's elliptic functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $\mathrm{dn}(u)$, of Jacobi's theta functions $\Theta(u)$, $H(u)$, $H_1(u)$, and $\Theta_1(u)$, and, finally, of Jacobi's zeta function, $\mathrm{zn}(u)$, (here we follow the notation of Carlson and Todd [6], in other references, like [30], Jacobi's zeta function is denoted by $Z(u)$). Let $k':=\sqrt{1-k^2}$ be the complementary modulus, let $K\equiv K(k)$ be the complete elliptic integral of the first kind, and define $K'\equiv K'(k):=K(k')$. For the definitions and many important properties of these functions, see, e.g., [5] and [16].

Let $a_1,a_2,a_3,a_4\in\mathbb{C}$ be four pairwise distinct points in the complex plane which satisfy inequality (29) and let the modulus k be given by

$$k^2:=\frac{(a_4-a_1)(a_3-a_2)}{(a_4-a_2)(a_3-a_1)}, \tag{30}$$

where we choose that branch of the square root in (30), for which $\operatorname{Re}k\geq0$. By (29), it suffices to consider the case

$$k\in D_k:=\bigl\{z\in\mathbb{C}:|z|\leq1,\operatorname{Re}z\geq0,z\notin\{0,1\}\bigr\} \tag{31}$$

in the following. Note that, by (31), the functions $K(k)$ and $K'(k)$ are single valued now (for a detailed discussion see [16, section 8.12]). We will need the function $\mathrm{sn}^2(u)$, which is an even elliptic function of order 2 with fundamental periods $2K$ and $2\mathrm{i}K'$, with a double zero at 0 and a double pole at $\mathrm{i}K'$. Further, for $k \in D_k$, let

$$\mathcal{P} := \big\{u \in \mathbb{C} : u = \lambda K + \mathrm{i}\lambda' K',\, (\lambda,\lambda') \in [0,1]\times[0,1] \cup (0,1)\times(-1,0)\big\} \tag{32}$$

be a "half" period parallelogram of $\mathrm{sn}^2(u)$ with respect to the modulus k. The mapping $\mathrm{sn}^2 : \mathcal{P} \to \overline{\mathbb{C}}$, $u \mapsto \mathrm{sn}^2(u)$, is bijective, hence for given $a_1, a_2, a_3, a_4 \in \mathbb{C}$ (pairwise distinct and satisfying (29)), there exists a unique $\rho \in \mathcal{P}$ such that

$$\mathrm{sn}^2(\rho) = \frac{a_4 - a_2}{a_4 - a_1}. \tag{33}$$

holds. Equation (33) is equivalent to

$$\mathrm{cn}^2(\rho) = \frac{a_2 - a_1}{a_4 - a_1} \quad \text{and} \quad \mathrm{dn}^2(\rho) = \frac{a_2 - a_1}{a_3 - a_1}, \tag{34}$$

respectively. On the other hand, given $k \in D_k$, $\rho \in \mathcal{P}$, and two of the four points a_1, a_2, a_3, a_4, say a_1 and a_2, by (30) and (33), a_3 and a_4 are given uniquely by

$$a_3 = \frac{a_2 - k^2 a_1\,\mathrm{sn}^2(\rho)}{\mathrm{dn}^2(\rho)}, \qquad a_4 = \frac{a_2 - a_1\,\mathrm{sn}^2(\rho)}{\mathrm{cn}^2(\rho)}. \tag{35}$$

Hence, each $(a_1, a_2, a_3, a_4) \in \mathbb{C}^4$, a_j pairwise distinct and satisfying (29) can be written in the form

$$(a_1, a_2, a_3, a_4) = \Big(a_1, a_2, \frac{a_2 - k^2 a_1\,\mathrm{sn}^2(\rho)}{\mathrm{dn}^2(\rho)}, \frac{a_2 - a_1\,\mathrm{sn}^2(\rho)}{\mathrm{cn}^2(\rho)}\Big). \tag{36}$$

In the following Characterization theorem [H8, Theorem 7], we gave necessary and sufficient conditions on k and ρ such that (a_1, a_2, a_3, a_4), given by (36), is a $\mathbf{T}_n$-tuple.

Theorem 11 (Peherstorfer and Schiefermayr [H8])**.** *Let $n \in \mathbb{N}$, let $a_1, a_2 \in \mathbb{C}$, $a_1 \neq a_2$, let $k \in D_k$ and $\rho \in \mathcal{P}$. The tuple (a_1, a_2, a_3, a_4) is a $\mathbf{T}_n$-tuple of the form* (36) *if and only if ρ is of the form*

$$\rho = \frac{mK + \mathrm{i}m'K'}{n}, \quad \textit{where} \quad m, m' \in \mathbb{Z}. \tag{37}$$

If ρ is of the form (37) *then the associated polynomials $\mathcal{T}_n$ and $\mathcal{U}_{n-2}$ from* (26) *are unique (up to their sign) and given by*

$$\mathcal{T}_n(z(u)) = \tfrac{1}{2}\big(\Omega(u) + \Omega(-u)\big) \tag{38}$$

and

$$\sqrt{\mathcal{H}(z(u))}\,\mathcal{U}_{n-2}(z(u)) = \tfrac{1}{2}\big(\Omega(u) - \Omega(-u)\big), \tag{39}$$

where $z(u)$ is defined by

$$z(u) := \frac{(a_2 - a_1)\,\mathrm{sn}^2(u)}{\mathrm{sn}^2(u) - \mathrm{sn}^2(\rho)} + a_1$$

and $\Omega(u)$ is defined by

$$\Omega(u) := \Big(\frac{H(u-\rho)}{H(u+\rho)}\Big)^n \exp\Big(\frac{-\mathrm{i}m'\pi u}{K}\Big).$$

Furthermore

$$\mathcal{T}_n^{-1}([-1,1]) = \big\{z(u) \in \mathbb{C} : u \in \mathcal{P}, |\Omega(u)| = 1\big\}.$$

The fact that condition (37) is necessary that an equation of the form (26) holds, where a_3, a_4 are given by (35), goes back to Zolotarev [31] already. He derived this condition when he studied the continued fractions of $\sqrt{\mathcal{H}(z)}$. Also the sufficiency part may be extracted from other work of Zolotarev [33], [34]. For a detailed proof, see [H8, Section 6]. The methods we used to prove the characterization theorem can be considered as an extension of those used in [31], [3], [10], and [20].

With the help of Theorem 11 it is easily proved that the set of all $\mathbf{T}_n$-tuples is dense in $\mathbb{C}^4$, see also [4]. The density for all real $\mathbf{T}_n$-tuples in $\mathbb{R}^4$ has been shown in [3], see also [22] and [29].

The above Characterization theorem also implies formulae for the point z^* introduced in (27) and for the logarithmic capacity of $\mathcal{T}_n^{-1}([-1,1])$, see [H8, Corollary 8]:

Theorem 12 (Peherstorfer and Schiefermayr [H8])**.** *Let ρ and $\mathcal{T}_n$ be given by* (37) *and* (38)*, respectively.*

(i) *Let $A := \mathcal{T}_n^{-1}([-1,1])$, then the logarithmic capacity of A is given by*

$$\operatorname{cap} A = \Big|\frac{K^2(a_4-a_3)(a_4-a_1)}{\pi^2(a_4-a_2)\Theta^4(\rho)\,\exp(\frac{\mathrm{i}m'\pi\rho}{nK})}\Big|.$$

(ii) *The zero z^* defined in* (27) *is given by*

$$z^* = a_2 + \frac{(a_2-a_1)\operatorname{sn}(\rho)}{2\operatorname{cn}(\rho)\operatorname{dn}(\rho)}\Big(\frac{\mathrm{i}m'\pi}{nK} + 2\operatorname{zn}(\rho)\Big). \tag{40}$$

4.2 Algebraic Solution

Using a recursive representation of the function $\operatorname{sn}^2(nu)$, $n\in\mathbb{N}$, see [26], we obtained an algebraic solution for the characterization of $\mathbf{T}_n$-tuples (a_1,a_2,a_3,a_4), see [H8, Section 4]:

Theorem 13 (Peherstorfer and Schiefermayr [H8])**.** *Let a_1,a_2,a_3,a_4 be four pairwise distinct points in $\mathbb{C}$ and let P_n and Q_n be given recursively by*

$$\begin{aligned} P_{\ell+1} &= (a_3-a_1)^2 Q_{\ell-1}\big[(a_4-a_1)P_\ell - (a_4-a_2)Q_\ell\big]^2,\\ Q_{\ell+1} &= (a_4-a_1)^2 P_{\ell-1}\big[(a_3-a_1)Q_\ell - (a_3-a_2)P_\ell\big]^2, \end{aligned}$$

where

$$P_1 := a_4-a_2,\quad Q_1 := a_4-a_1,\quad P_2 := 4(a_4-a_2)(a_3-a_1),\quad Q_2 := (a_1+a_2-a_3-a_4)^2.$$

Then (a_1,a_2,a_3,a_4) is a $\mathbf{T}_n$-tuple if and only if

$$(a_4-a_1)(a_3-a_2)P_n - (a_4-a_2)(a_3-a_1)Q_n = 0.$$

In [H6], we obtained an *explicit* polynomial equation (in form of some determinants) for the points $\{a_1,a_2,a_3,a_4\}$ to be a $\mathbf{T}_n$-tuple. Moreover, explicit polynomial equations for the extremal points z_j of $\mathcal{T}_n$ on $\mathcal{T}_n^{-1}([-1,1])$ are given, i.e., for those points z_j, for which $\mathcal{T}_n(z_j)=\pm1$. Note that with the help of these extremal points and the points $\{a_1,a_2,a_3,a_4\}$, the corresponding polynomial $\mathcal{T}_n$ can represented in a simple way.

4.3 Three Special Cases

In this section, we focus on the case of two arcs symmetric with respect to the real line. We distinguish three cases. First, when all endpoints a_1, a_2, a_3, a_4 are real, investigated by Zolotarev [32] and in detail by Achieser [3] and then, when two points are real and two points are complex conjugate, again studied by Zolotarev [32], Achieser [1] and Lebedev [17]. So far the two cases have been treated separately. Here we show how to obtain both cases and the case of two pairs of complex conjugate points by a unified approach.

4.3.1 Inverse Polynomial Images which Consist of Two Intervals

For the simplest case, $a_1, a_2, a_3, a_4 \in \mathbb{R}$, one obtains Achieser's result for two intervals [3].

Theorem 14. *Let $n \in \mathbb{N}$, let $a_1, a_2 \in \mathbb{R}$, $a_1 < a_2$, let $k \in D_k$, and let $\rho \in \mathcal{P}$. The tuple (a_1, a_2, a_3, a_4) given in the form* (36) *is a real $\mathbf{T}_n$-tuple, i.e., $a_3, a_4 \in \mathbb{R}$, with $a_1 < a_2 < a_3 < a_4$, if and only if $k \in (0,1)$ and $\rho = \frac{mK}{n}$, $m \in \{1, 2, \ldots, n-1\}$. The associated polynomials $\mathcal{T}_n$ and $\mathcal{U}_{n-2}$ given by* (38) *and* (39) *have real coefficients and*

$$\mathcal{T}_n^{-1}([-1,1]) = [a_1, a_2] \cup [a_3, a_4]. \tag{41}$$

Note that z^* from (40) *satisfies $z^* \in (a_2, a_3)$.*

For $n = 5$, $(a_1, a_2, a_3, a_4) = (-1, 0.068\ldots, 0.522\ldots, 1)$ is a real $\mathbf{T}_n$-tuple, where $k = 0.8$ and $m = 2$. Fig. 3 shows the graph of the associated polynomial $\mathcal{T}_n$ (left plot) and the sets $\mathcal{T}_n^{-1}([-1,1]) = [a_1, a_2] \cup [a_3, a_4]$ (thick line) and $\mathcal{T}_n^{-1}(\mathbb{R})$ (dotted line) on the right hand side.

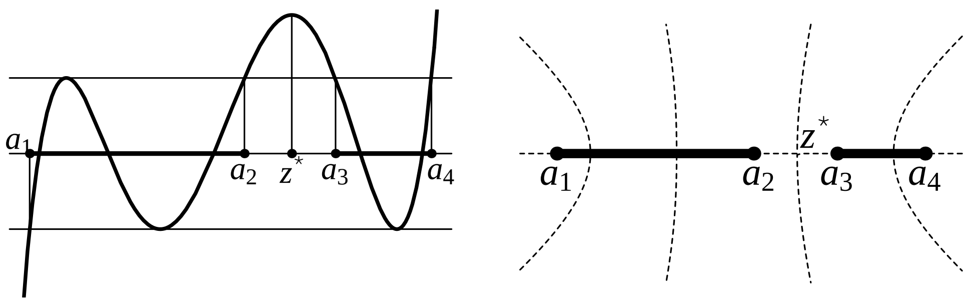

Figure 3: Associated polynomial and arcs for the real case

4.3.2 Inverse Polynomial Images which Consist of an Interval and an Arc

For the case, when two of the endpoints are real (without loss of generality we fixed them at ± 1) and the other two are complex conjugate, it turned out that the corresponding modulus k lies on the unit disk, see [H8, Section 5.2]:

Theorem 15 (Peherstorfer and Schiefermayr [H8])**.** *Let $n \in \mathbb{N}$, let $a_1 = -1$, $a_2 = 1$, let $k \in D_k$, and let $\rho \in \mathcal{P}$. The tuple (a_1, a_2, a_3, a_4) given in the form* (36) *is a $\mathbf{T}_n$-tuple with the properties $a_3 = \overline{a}_4$ and $\operatorname{Im} a_3 > 0$ if and only if $k = e^{\mathrm{i}\varphi}$, $\varphi \in (0, \frac{\pi}{2})$, and $\rho = (mK + \mathrm{i}m'K')/n$ with $m, m' \in \mathbb{Z}$ and $m = 0$ or $2m' + m = 0$ or $2m' + m = 2n$.*

Using the transformation $k \mapsto \frac{1}{2}(\sqrt{k} + \frac{1}{\sqrt{k}})$ and transforming Jacobi's elliptic and theta functions (not all these transformation formulas were available in the literature), we arrived at the following characterization, see [H8, Section 5.2]. Parts of this characterization can be found in Achieser [1] and Lebedev [17].

Theorem 16 (Peherstorfer and Schiefermayr [H8])**.** *For $0 < k < 1$ and $0 < \lambda < \frac{1}{2}$, consider the mapping*

$$\alpha + \mathrm{i}\beta \equiv \alpha(k,\lambda) + \mathrm{i}\beta(k,\lambda) = \frac{\mathrm{cn}(2\lambda K)}{\mathrm{dn}^2(2\lambda K)} + \mathrm{i}\,\frac{kk'\mathrm{sn}^2(2\lambda K)}{\mathrm{dn}^2(2\lambda K)}. \tag{42}$$

(i) *The mapping* (42) *is bijective from $(0,1) \times (0, \frac{1}{2})$ onto $\{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$.*

(ii) *The tuple $(a_1, a_2, a_3, a_4) = (-1, 1, \alpha + \mathrm{i}\beta, \alpha - \mathrm{i}\beta)$ is a $\mathbf{T}_n$-tuple if and only if $\lambda = \frac{m}{n}$, where $m \in \mathbb{N}$, i.e., if and only if λ is rational with denominator n.*

(iii) *The associated polynomial $\mathcal{T}_n$ is given by*

$$\mathcal{T}_n(z(u)) = \frac{1}{2}\Big(\Omega^n(u) + \frac{1}{\Omega^n(u)}\Big), \tag{43}$$

where

$$\Omega(u) := \frac{H(u - \lambda K)\,\Theta_1(u - \lambda K)}{H(u + \lambda K)\,\Theta_1(u + \lambda K)} \tag{44}$$

and

$$z(u) := \frac{\mathrm{cn}(2u)\,\mathrm{cn}(2\lambda K) - 1}{\mathrm{cn}(2u) - \mathrm{cn}(2\lambda K)}. \tag{45}$$

(iv) *The point z^* defined by relation* (27) *is given by*

$$z^* = 1 + \frac{1}{\mathrm{dn}(2\lambda K)}\big(\mathrm{cn}(2\lambda K) - 1 + 2\,\mathrm{sn}(2\lambda K)\,\mathrm{zn}(\lambda K)\big) \tag{46}$$

and $z^* > 0$ holds.

(v) *The inverse image $\mathcal{T}_n^{-1}([-1,1])$ is given by*

$$\mathcal{T}_n^{-1}([-1,1]) = \big\{z(u) \in \mathbb{C} : u \in \mathbb{C}, |\Omega(u)| = 1\big\} \tag{47}$$

and consists of the interval $[-1,1]$ and a Jordan arc symmetric with respect to the real line with the endpoints a_3 and a_4. We will denote this Jordan arc by $\widetilde{a_3a_4}$.

(vi) *If $z^* \le 1$ then the arc $\widetilde{a_3a_4}$ crosses the interval $[-1,1]$ at z^*.*

(vii) *If $z^* > 1$, then $\widetilde{a_3a_4}$ and $[-1,1]$ are noncrossing.*

Concerning the number of extremal points on the two arcs, we proved the following, see [H3, Theorem 4]:

Theorem 17 (Schiefermayr [H3])**.** *Let $0 < k < 1$, $0 < \lambda < \frac{1}{2}$, $\lambda = \frac{m}{n}$, $m, n \in \mathbb{N}$, $0 < m < \frac{n}{2}$, let $z(u)$ and $\Omega(u)$ be defined by* (45) *and* (44)*, respectively, and let $\mathcal{T}_n(z(u))$ defined by* (43)*.*

(i) *If $z^* \le 1$, i.e., the arcs $\widetilde{a_3a_4}$ and $[-1,1]$ are crossing at the point z^*, then $\mathcal{T}_n(z)$ has at least $n - 2m + 1$ extremal points on $[-1,1]$.*

(ii) *If $z^* > 1$, i.e., the arcs $\widetilde{a_3a_4}$ and $[-1,1]$ are noncrossing, then $\mathcal{T}_n(z)$ has exactly $n-2m+1$ extremal points on $[-1,1]$ and exactly $2m+1$ extremal points on $\widetilde{a_3a_4}$.*

In Fig. 4, for $n=8$ and $\lambda = \frac{2}{8}$ (i.e. $m=2$), the inverse images $\mathcal{T}_n^{-1}(\mathbb{R})$ (dotted line) and $\mathcal{T}_n^{-1}([-1,1])$ (solid line) are plotted for the cases $z^* < 1$ (picture above), $z^* = 1$ (picture in the middle) and $z^* > 1$ (picture below), respectively, see Theorem 16 (ix) and (x). Note that the extremal points of $\mathcal{T}_n$ on $\mathcal{T}_n^{-1}([-1,1])$ are the endpoints of the two arcs $[-1,1]$ and $\widetilde{a_3a_4}$ and the crossing points of the dotted and the solid lines, see Theorem 17. It is remarkable that in the borderline case $z^*=1$, the Jordan arc $\widetilde{a_3a_4}$ is *not* analytic.

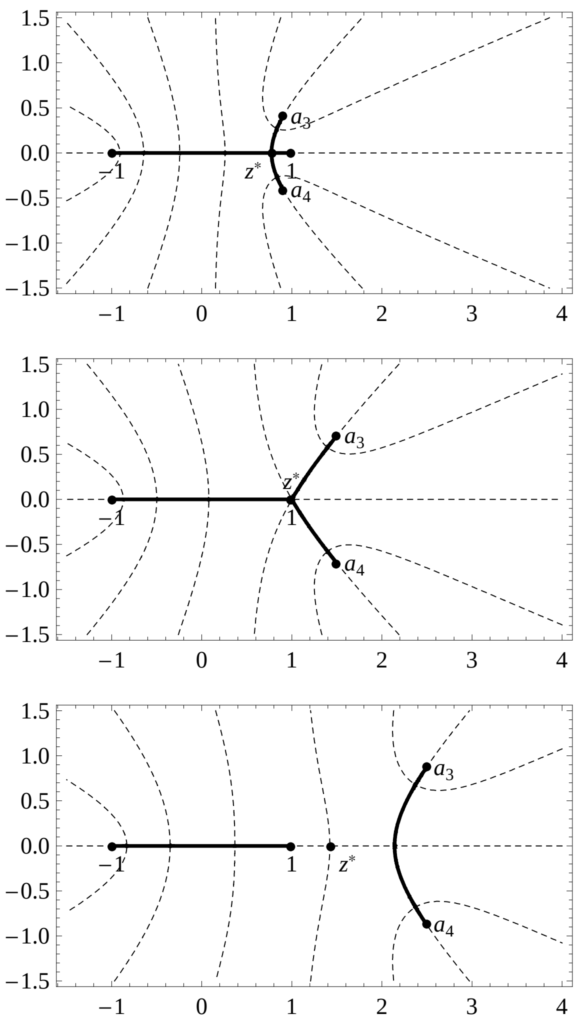

Figure 4: Inverse images $\mathcal{T}_n^{-1}(\mathbb{R})$ (dotted line) and $\mathcal{T}_n^{-1}([-1,1])$ (solid line)

The next theorem gives a density result, see [H3, Theorem 5].

Theorem 18 (Schiefermayr [H3])**.** *Let $\alpha, \beta > 0$ and $n \in \mathbb{N}$ be given. Then there exist $\alpha^*, \beta^* > 0$ such that $(a_1,a_2,a_3,a_4) = (-1,1,\alpha^*+\mathrm{i}\beta^*, \alpha^*-\mathrm{i}\beta^*)$ is a $\mathbf{T}_n$-tuple and*

$$|(\alpha+\mathrm{i}\beta) - (\alpha^*+\mathrm{i}\beta^*)| \le \frac{A}{n}$$

and A is a constant independent of n.

By Theorem 16, the connectivity of the two arcs $[-1,1]$ and $\widetilde{a_3a_4}$ can be characterized with the help of z^*. For this reason, the next theorem [H3, Theorem 6] is important:

Theorem 19 (Schiefermayr [H3]). *For fixed λ, $0<\lambda<\frac{1}{2}$, the point z^* given in (46), as a function of the modulus k, $0<k<1$, is strictly monotone increasing with $z^*\to\cos(\lambda\pi)$ as $k\to 0$ and $z^*\to\infty$ as $k\to 1$.*

By Theorem 19, for each $\lambda\in(0,\frac{1}{2})$, there exists a unique modulus $k^*\equiv k^*(\lambda)$, for which $z^*=1$, Thus, for $k\leq k^*$, the arc $\widetilde{a_3a_4}$ is crossing the interval $[-1,1]$ (at z^*), and, for $k>k^*$, $\widetilde{a_3a_4}$ and $[-1,1]$ are noncrossing. Compare Fig. 4, which shows the inverse images $\mathcal{T}_n^{-1}(\mathbb{R})$ (dotted line) and $\mathcal{T}_n^{-1}([-1,1])$ (solid line) for $n=8$, $\lambda=2/8$ and $k=0.7<k^*$ (picture above), $k^*=0.942809\ldots$ (picture in the middle) and $k=0.99>k^*$ (picture below), respectively.

Fig. 5 shows that parametric curve $\alpha(\lambda,k^*(\lambda))+\mathrm{i}\beta(\lambda,k^*(\lambda))$, on which $z^*=1$, i.e., for all points $a_3=\alpha+\mathrm{i}\beta$, $\alpha,\beta>0$, lying on the left hand side [right hand side] of this curve, the corresponding arc $\widetilde{a_3a_4}$ is crossing [not crossing] the interval $[-1,1]$.

We were able to prove [H3, Theorem 7] that the condition $\alpha\leq 1$ is a sufficient condition for $a_3=\alpha+\mathrm{i}\beta$ such that the two arcs $\widetilde{a_3a_4}$ and $[-1,1]$ are crossing each other.

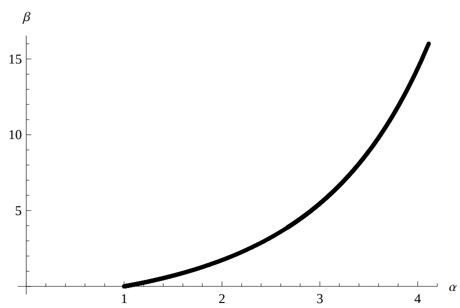

Figure 5: Parametric curve $\alpha(\lambda,k^*)+\mathrm{i}\beta(\lambda,k^*)$, on which $z^*=1$

4.3.3 Inverse Polynomial Images which Consist of Two Jordan Arcs Symmetric with respect to the Real Line

Finally, we studied the case when the $\mathbf{T}_n$-tuple (a_1,a_2,a_3,a_4) consists of two pairs of complex conjugate numbers (without loss of generality let us fix one pair at $\pm\mathrm{i}$). Since it turned out that the coefficients of the associated polynomial $\mathcal{T}_n$ are either real or purely imaginary, $\mathcal{T}_n^{-1}([-1,1])$ consists of two arcs symmetric with respect to the real line, see Fig. 6. We proved the following characterization [H8, Section 5.3]:

Theorem 20 (Peherstorfer and Schiefermayr [H8]). *Let $n\in\mathbb{N}$, let $a_1=\mathrm{i}$, $a_2=\mathrm{i}$, let $k\in D_k$, and let $\rho\in\mathcal{P}$. The tuple (a_1,a_2,a_3,a_4) given in the form (36) is a $\mathbf{T}_n$-tuple with $\operatorname{Re}a_3>0$, $\operatorname{Im}a_3<0$ and $a_4=\overline{a}_3$, if and only if $k\in(0,1)$ and*

$$\rho=\frac{mK+\mathrm{i}\,m'K'}{n},\quad n\ \textit{even},\quad m'=\tfrac{n}{2},\quad m\in\{1,2,\ldots,n-1\}. \tag{48}$$

Finally, we obtained a result concerning the shape of the two arcs depending on the parameter k, see [H8, Section 5.3].

Theorem 21 (Peherstorfer and Schiefermayr [H8]). *Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$ be as in Theorem 20, suppose that ρ is of the form* (48)*, i.e.,* (a_1, a_2, a_3, a_4) *is a* $\mathbf{T}_n$*-tuple, and let* $\mathcal{T}_n$ *be the associated polynomial. Then* $\mathcal{T}_n^{-1}([-1,1])$ *consists of two Jordan arcs symmetric with respect to the real line. Further, there exists a* $k^* \in (0,1)$*, such that the following three statements hold:*

(i) *for* $k \in (0, k^*)$*,* $\mathcal{T}_n^{-1}([-1,1])$ *consists of two Jordan arcs, one moving from* a_1 *to* a_4 *and one moving from* a_2 *to* a_3*, where both arcs are not crossing the real line;*

(ii) *for* $k = k^*$*,* $\mathcal{T}_n^{-1}([-1,1])$ *consists of two Jordan arcs which cross each other at the point* $z^* \in \mathbb{R}$ *given in* (40)*;*

(iii) *for* $k \in (k^*, 1)$*,* $\mathcal{T}_n^{-1}([-1,1])$ *consists of two Jordan arcs, one moving from* a_1 *to* a_2*, crossing the real line at* $z_1 \in \mathbb{R}$ *and one moving from* a_3 *to* a_4*, crossing the real line at* $z_2 \in \mathbb{R}$*, where* $z_1 < z_2$*.*

The proof of Theorem 21 is based on some monotonicity results for Jacobi's theta function $\Theta(u) \equiv \Theta(u,k)$ with respect to the corresponding modulus k proved in [H7].

For $n = 6$, $a_1 = -1 + 2\mathrm{i}$, $a_2 = -1 - 2\mathrm{i}$, $m = 2$, $m' = n/2 = 3$, according to Theorem 21, Fig. 6 shows the three possible shapes of $\mathcal{T}_n^{-1}([-1,1])$, where for the modulus $k \in (0,1)$ we have set $k = 0.6$, $k = k^* = 0.734465\ldots$, and $k = 0.8$. The plots show the endpoints a_1, a_2, a_3, a_4, the point z^*, and the sets $\mathcal{T}_n^{-1}([-1,1])$ (fat line) and $\mathcal{T}_n^{-1}(\mathbb{R})$ (dotted line).

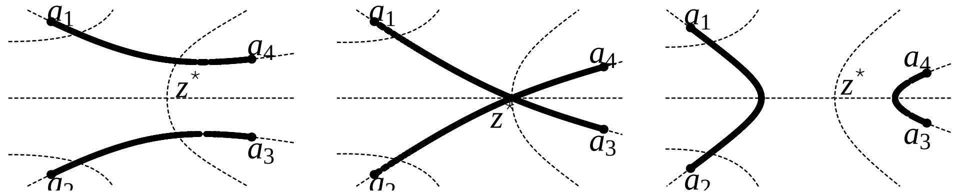

Figure 6: The three possible shapes of $\mathcal{T}_n^{-1}([-1,1])$ according to Theorem 21.

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Appendix A: List of Publications

[P1] Klaus Schiefermayr, A lower bound for the norm of the minimal residual polynomial, *Constructive Approximation* (2010).

[P2] Klaus Schiefermayr, Estimates for the asymptotic convergence factor of two intervals, *Journal of Computational and Applied Mathematics* (2010).

[P3] Klaus Schiefermayr, Inverse polynomial images consisting of an interval and an arc, *Computational Methods and Function Theory* **9** (2009), 407–420.

[P4] Klaus Schiefermayr, A lower bound for the minimum deviation of the Chebyshev polynomial on a compact real set, *East Journal on Approximations* **14** (2008), 223–233.

[P5] Klaus Schiefermayr, An upper bound for the logarithmic capacity of two intervals, *Complex Variables and Elliptic Equations* **53** (2008), 65–75.

[P6] Klaus Schiefermayr, Inverse polynomial images which consists of two Jordan arcs – an algebraic solution, *Journal of Approximation Theory* **148** (2007), 148–157.

[P7] Klaus Schiefermayr and Josef Weichbold, A scheduling problem for several parallel servers, *Mathematical Methods of Operations Research* **66** (2007), 127–148.

[P8] Josef Weichbold and Klaus Schiefermayr, The optimal control of a general tandem queue, *Probability in the Engineering and Informational Sciences* **20** (2006), 307–327.

[P9] Klaus Schiefermayr and Josef Weichbold, A complete solution for the optimal stochastic scheduling of a two-stage tandem queue with two flexible servers, *Journal of Applied Probability* **42** (2005), 778–796.

[P10] Klaus Schiefermayr, Some new properties of Jacobi's theta functions, *Journal of Computational and Applied Mathematics* **178** (2005), 419–424.

[P11] Franz Peherstorfer and Klaus Schiefermayr, Description of inverse polynomial images which consist of two Jordan arcs with the help of Jacobi's elliptic functions, *Computational Methods and Function Theory* **4** (2004), 355–390.

[P12] Klaus Schiefermayr, Random walks with similar transition probabilities, *Journal of Computational and Applied Mathematics* **153** (2003), 423–432.

[P13] Franz Peherstorfer and Klaus Schiefermayr, On the connection between minimal polynomials on arcs and on intervals, *Functions, series, operators* (Budapest, 1999), 339–356, János Bolyai Math. Soc., Budapest, 2002.

[P14] Dinesh S. Bhoj and Klaus Schiefermayr, Approximations to the distribution of weighted combination of independent probabilities, *Journal of Statistical Computation and Simulation* **68** (2001), 153–159.

[P15] Franz Peherstorfer and Klaus Schiefermayr, Description of extremal polynomials on several intervals and their computation. I, *Acta Mathematica Hungarica* **83** (1999), 27–58.

[P16] Franz Peherstorfer and Klaus Schiefermayr, Description of extremal polynomials on several intervals and their computation. II, *Acta Mathematica Hungarica* **83** (1999), 59–83.

[P17] Franz Peherstorfer and Klaus Schiefermayr, Explicit generalized Zolotarev polynomials with complex coefficients. II, *East Journal on Approximations* **3** (1997), 473–483.

Appendix B: List of Talks

[T1] *Inequalities for the deviation of minimal residual polynomials and inverse polynomial images*, 13th International Conference Approximation Theory, San Antonio, Texas, USA, 2010.

[T2] *Inverse polynomial images which consists of two Jordan arcs and sets of minimal logarithmic capacity*, Conference on Computational Methods and Function Theory, Bilkent University, Ankara, Turkey, 2009.

[T3] *Estimates for the logarithmic capacity and the minimum deviation of Chebyshev polynomials*, International Workshop on Orthogonal Polynomials and Approximation Theory, Madrid/Leganes, Spain, 2008.

[T4] *Upper and lower bounds for the logarithmic capacity and the minimum deviation of several intervals*, Conference on Special Functions, Information Theory and Mathematical Physics, Granada, Spain, 2007.

[T5] *Inverse polynomial images which consists of two Jordan arcs - an algebraic solution*, 12th International Conference Approximation Theory, San Antonio, Texas, USA, 2007.

[T6] *A scheduling problem with m identical parallel servers and two different types of jobs*, International Conference of Operations Research, Tilburg, Netherlands, 2004.

[T7] *Description of inverse polynomial images which consists of two Jordan arcs with the help of Jacobis elliptic and theta functions*, Seventh International Symposium on Orthogonal Polynomials, Conference on Special Functions and Applications, Copenhagen, 2003.

[T8] *Optimal stochastic scheduling of a two-stage tandem queue with two flexible servers*, First Madrid Conference on Queueing Theory, Madrid, 2002.

[T9] *Random Walks with similar transition probabilities*, Sixth International Symposium on Orthogonal Polynomials, Special Functions and Applications, Rome, 2001.

[T10] *Approximations to the distribution of the sum of Erlang-distributed variates*, Jahrestagung Deutsche Mathematiker-Vereinigung (DMV), Dresden, 2000.

[T11] *On the connection between minimal polynomials on arcs and on intervals*, Functions, Series, Operators: Alexits Memorial Conference, Budapest, 1999.