A Density Result Concerning Inverse Polynomial Images*

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Abstract

In this paper, we consider polynomials of degree n, for which the inverse image of $[-1,1]$ consists of two Jordan arcs. We prove that the four endpoints of these arcs form an $\mathcal{O}(1/n)$-net in the complex plane.

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1 Introduction and Main Result

Let $\mathbb{P}_n$ be the set of all polynomials of degree n with complex coefficients. For a polynomial $\mathcal{T}_n \in \mathbb{P}_n$, consider the inverse image of $[-1,1]$ defined by

$$\mathcal{T}_n^{-1}([-1,1]) := \big\{ z \in \mathbb{C} : \mathcal{T}_n(z) \in [-1,1] \big\}. \tag{1}$$

Inverse polynomial images are interesting for instance in approximation theory, since each polynomial $\mathcal{T}_n \in \mathbb{P}_n$ is (suitably normed) the Chebyshev polynomial (i.e. the minimal polynomial with respect to the supremum norm) of degree n on its inverse image $\mathcal{T}_n^{-1}([-1,1])$; see [5], [7], [8], or [4].

In the following, we will need the notion of Jordan arcs. A set $\big\{\gamma(t) \in \overline{\mathbb{C}} : t \in [0,1]\big\}$ is called a *Jordan arc* if $\gamma : [0,1] \to \overline{\mathbb{C}}$ is continuous and $\gamma : [0,1) \to \overline{\mathbb{C}}$ is injective.

It is well known that $\mathcal{T}_n^{-1}([-1,1])$ is the union of n Jordan arcs. The number of Jordan arcs can be reduced for some polynomials $\mathcal{T}_n$: the inverse image $\mathcal{T}_n^{-1}([-1,1])$ consists of one Jordan arc if and only if $\mathcal{T}_n(z) = T_n(az+b)$, where T_n is the classical Chebyshev polynomial of the first kind, i.e. $T_n(z) = \cos(n \arccos z)$; see [12, Cor. 1]. In this case the inverse image is an interval in the complex plane (which of course can be seen as the union of n intervals). In other words, the case of one Jordan arc is trivial.

In this note, we are interested in polynomials with an inverse image consisting of *two Jordan arcs*. Given four pairwise distinct points $a_1, a_2, a_3, a_4 \in \mathbb{C}$ in the complex plane, define

$$\mathcal{H}_4(z) := (z-a_1)(z-a_2)(z-a_3)(z-a_4). \tag{2}$$

Then the following characterization theorem holds; see [12, Thm. 1] or [10, Thm. 3].

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Theorem 1. *Let $\mathcal{T}_n(z) = \tau z^n + \ldots \in \mathbb{P}_n$ be any polynomial of degree n. Then $\mathcal{T}_n^{-1}([-1,1])$ consists of two (but not less than two) Jordan arcs with endpoints a_1, a_2, a_3, a_4 if and only if $\mathcal{T}_n^2 - 1$ has exactly 4 pairwise distinct zeros a_1, a_2, a_3, a_4 of odd multiplicity, i.e., if and only if $\mathcal{T}_n$ satisfies a polynomial equation of the form*

$$\mathcal{T}_n^2(z) - 1 = \mathcal{H}_4(z)\,\mathcal{U}_{n-2}^2(z) \tag{3}$$

with $\mathcal{U}_{n-2}(z) = \tau z^{n-2} + \ldots \in \mathbb{P}_{n-2}$ and $\mathcal{H}_4$ given in (2). In this case, the tuple $(a_1, a_2, a_3, a_4) \in \mathbb{C}^4$ is called a $\mathbf{T}_n$-tuple.

Condition (3) implies that $\mathcal{T}_n^{-1}([-1,1])$ consists of two Jordan arcs, which are not necessarily analytic. Concerning the minimum number of *analytic* Jordan arcs, we refer to [12, Thm. 3].

Now, we are able to state the main result. Roughly stated it says that all $\mathbf{T}_n$-tuples (a_1, a_2, a_3, a_4) form an $\mathcal{O}(\frac{1}{n})$-net in the complex plane.

Theorem 2. *Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$ be four pairwise distinct points in the complex plane. Then there exist $\tilde{a}_2, \tilde{a}_3 \in \mathbb{C}$ such that $(a_1, \tilde{a}_2, \tilde{a}_3, a_4)$ is a $\mathbf{T}_n$-tuple and*

$$|a_2 - \tilde{a}_2| \leq \frac{C_1}{n} \qquad \textit{and} \qquad |a_3 - \tilde{a}_3| \leq \frac{C_2}{n}$$

hold for $n \geq N$, where C_1, C_2, N depend only on a_1, a_2, a_3, a_4 but do not depend on n.

Remark. (i) The proof of Theorem 2 is given in the next section.

(ii) The values C_1 and C_2 can be expressed using Jacobian elliptic functions; see (26) and (27) in the proof of Theorem 2, respectively.

(iii) For the real case, i.e. $a_1, a_2, a_3, a_4 \in \mathbb{R}$, the density result of Theorem 2 was proved long time ago by Achieser [1].

(iv) For the special case $a_1, a_2 \in \mathbb{R}$, $a_3 \in \mathbb{C} \setminus \mathbb{R}$, $a_4 = \overline{a}_3$, the density result of Theorem 2 is proved in [11].

(v) For the case of ℓ real intervals, a density result similar to that of Theorem 2 was proved independently about the same time by Bogatyrev [2], Peherstorfer [9], and Totik [14].

2 Proof of the Main Result

The proof of the main result is managed with the help of the characterization of a $\mathbf{T}_n$-tuple using Jacobian elliptic and theta functions; see [10]. To this end, let us briefly recall some definitions. For an introduction to elliptic functions and integrals, we refer to [3] and [6].

Let $k \in D_k$, D_k defined in (6), be the modulus of the Jacobian elliptic functions $\operatorname{sn}(u) = \operatorname{sn}(u,k)$, $\operatorname{cn}(u) = \operatorname{cn}(u,k)$, and $\operatorname{dn}(u) = \operatorname{dn}(u,k)$. Let k', defined by ${k'}^2 := 1 - k^2$, be the complementary modulus, let $K = K(k)$ be the complete elliptic integral of the first kind and let $K' = K'(k) := K(k')$.

Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$ be four pairwise distinct complex points and define the modulus k by

$$k^2 := \frac{(a_4 - a_1)(a_3 - a_2)}{(a_4 - a_2)(a_3 - a_1)}. \tag{4}$$

By optionally exchanging some of the a_i's, it is always possible to get a tuple (a_1, a_2, a_3, a_4) which satisfies

$$|a_4 - a_1| \cdot |a_3 - a_2| \leq |a_4 - a_2| \cdot |a_3 - a_1|, \tag{5}$$

i.e., $|k^2| \leq 1$, and for which $k^2 \notin \mathbb{R}^-$. Obviously $k^2 = 0$ if and only if $(a_1 = a_4 \vee a_2 = a_3)$ and $k^2 = 1$ if and only if $(a_1 = a_2 \vee a_3 = a_4)$. In order to get k, we choose the branch of the square root in (4) for which $\mathrm{Re}(k) > 0$. Therefore, we have to consider the case

$$k \in D_k := \left\{ u \in \mathbb{C} : |u| \leq 1, \mathrm{Re}(u) > 0, u \neq 1 \right\} \tag{6}$$

in the following. For $k \in D_k$, the functions $K = K(k)$ and $K' = K'(k)$ are single valued (for a detailed discussion see [6, Sect. 8.12]). For the complementary modulus k', by (4) and the relation ${k'}^2 = 1 - k^2$, we get

$${k'}^2 = \frac{(a_4 - a_3)(a_2 - a_1)}{(a_4 - a_2)(a_3 - a_1)}. \tag{7}$$

Further, we will need the function $\mathrm{sn}^2(u)$, which is an even elliptic function of order 2 with fundamental periods $2K$ and $2\mathrm{i}K'$, with a double zero at 0 and a double pole at $\mathrm{i}K'$. Further, for $k \in D_k$, let

$$\mathcal{P} := \left\{ \mu K + \mathrm{i}\mu' K' : 0 \leq \mu, \mu' \leq 1 \vee (0 < \mu < 1 \wedge -1 < \mu' < 0) \right\} \tag{8}$$

be a "half" period parallelogram of $\mathrm{sn}^2(u)$ with respect to the modulus k. Note that $\mathrm{i}K'/K \notin \mathbb{R}$. By the above mentioned properties of $\mathrm{sn}^2(u)$, the mapping $\mathrm{sn}^2 : \mathcal{P} \to \overline{\mathbb{C}}$, $u \mapsto \mathrm{sn}^2(u)$, is bijective; hence for given $a_1, a_2, a_3, a_4 \in \mathbb{C}$, there exists a unique $\varrho = \lambda K + \mathrm{i}\lambda' K' \in \mathcal{P}$ with $\lambda, \lambda' \in \mathbb{R}$ such that

$$\mathrm{sn}^2(\varrho) = \mathrm{sn}^2(\lambda K + \mathrm{i}\lambda' K') = \frac{a_4 - a_2}{a_4 - a_1}. \tag{9}$$

Note that $\varrho = 0, K, K + \mathrm{i}K', \mathrm{i}K'$ is equivalent to $\mathrm{sn}^2(\varrho) = 0, 1, 1/k^2, \infty$, respectively, and that $(a_4 - a_2)/(a_4 - a_1) = 0, 1, 1/k^2, \infty$ is possible only if $a_4 = a_2$, $a_2 = a_1$, $a_2 = a_1$, $a_4 = a_1$, respectively. Thus, since a_1, a_2, a_3, a_4 are pairwise distinct, we have $\varrho \notin \{0, K, K + \mathrm{i}K', \mathrm{i}K'\}$.

In the following characterization theorem [10, Thm. 7], a necessary and sufficient condition is given such that (a_1, a_2, a_3, a_4) is a $\mathbf{T}_n$-tuple.

Theorem 3. *Let $n \in \mathbb{N}$, let $a_1, a_2, a_3, a_4 \in \mathbb{C}$ be pairwise distinct and satisfy* (5)*, and let $k \in D_k$ and $\varrho \in \mathcal{P}$ be defined by* (4) *and* (8)*, respectively. Then (a_1, a_2, a_3, a_4) is a $\mathbf{T}_n$-tuple if and only if ϱ is of the form*

$$\varrho = \tfrac{m}{n} K + \mathrm{i}\, \tfrac{m'}{n} K', \quad \textit{where} \quad m, m' \in \mathbb{Z}. \tag{10}$$

In Figure 1, the set $\mathcal{P}$, defined in (8), and all points $\varrho \in \mathcal{P}$ of the form (10) (where $n = 6$ was chosen) are illustrated.

If (10) holds, i.e. if (a_1, a_2, a_3, a_4) is a $\mathbf{T}_n$-tuple, then the corresponding polynomials $\mathcal{T}_n(z)$, $\mathcal{U}_{n-2}(z)$, and $\mathcal{H}_4(z)$ of Theorem 1 can be represented with a certain conformal mapping and with the help of Jacobi's elliptic and theta functions; for details we refer to [10].

Before we start with the proof of Theorem 2, let us state an inequality for the elliptic sine function, which will be crucial for the proof.

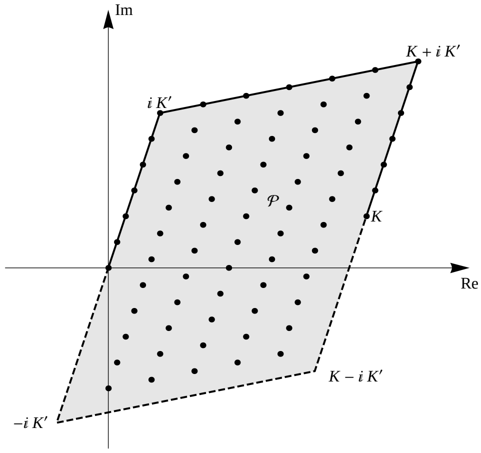

Figure 1: Illustration of the parallelogram $\mathcal{P}$

Lemma 1. *Let $k \in D_k$, and $u \in \mathbb{C}$, $|u| \le \frac{\pi}{4}$. Then*

$$|\operatorname{sn}(u)| \le \tan|u| \le \tfrac{4}{\pi}|u|.$$

Proof. A more general version of the first inequality is proved in [13]. The second inequality follows immediately from $f(x) := \tan x - \frac{4}{\pi}x \le 0$, $0 \le x \le \frac{\pi}{4}$, since $f(0) = f(\frac{\pi}{4}) = 0$ and $f''(x) = 2\sin x/\cos^3 x > 0$ for $0 \le x \le \frac{\pi}{4}$. □

Proof of Theorem 2. Let $n \in \mathbb{N}$ and let us assume that a_1, a_2, a_3, a_4 satisfy inequality (5) (which is always possible by a reordering of the a_j). Let k be defined by (4), let $\varrho \in \mathcal{P}$ be defined by equation (9) and let $\lambda, \lambda' \in \mathbb{R}$ be uniquely defined by

$$\varrho = \lambda K + \mathrm{i}\lambda' K'. \tag{11}$$

By (4), (7), and (9),

$$a_2 = a_4 - (a_4 - a_1)\operatorname{sn}^2(\varrho) \tag{12}$$

and

$$a_3 = \frac{a_4 - k'^2 a_1}{k^2} \cdot \frac{a_2 + \dfrac{k^2 a_1 a_4}{k'^2 a_1 - a_4}}{a_2 + \dfrac{k'^2 a_4 - a_1}{k^2}} = A_1 \cdot \frac{a_2 + A_2}{a_2 + A_3}, \tag{13}$$

where

$$A_1 := \frac{a_4 - k'^2 a_1}{k^2}, \quad A_2 := \frac{k^2 a_1 a_4}{k'^2 a_1 - a_4}, \quad A_3 := \frac{k'^2 a_4 - a_1}{k^2}. \tag{14}$$

Clearly there exist integers $m, m' \in \mathbb{Z}$ such that

$$\left|\frac{m}{n} - \lambda\right| \le \frac{1}{n} \qquad \text{and} \qquad \left|\frac{m'}{n} - \lambda'\right| \le \frac{1}{n}. \tag{15}$$

Let us remark that the integers m, m' can be chosen such that (cf. Figure 1)

$$\varrho \pm \tilde{\varrho} \notin \{\nu K + \mathrm{i}\,\nu' K' : \nu, \nu' \in \mathbb{Z}\}, \tag{16}$$

where $\tilde{\varrho}$ is defined by

$$\tilde{\varrho} := \tfrac{m}{n}\,K + \mathrm{i}\,\tfrac{m'}{n}K'. \tag{17}$$

Note that (16) implies that none of the points $\varrho \pm \tilde{\varrho}$ or $\frac{1}{2}(\varrho \pm \tilde{\varrho})$ is a pole of $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, or $\mathrm{dn}(u)$. Define

$$\tilde{a}_2 := a_4 - (a_4 - a_1)\,\mathrm{sn}^2(\tilde{\varrho}) \tag{18}$$

and

$$\tilde{a}_3 := \frac{a_4 - k'^2 a_1}{k^2} \cdot \frac{\tilde{a}_2 + \dfrac{k^2 a_1 a_4}{k'^2 a_1 - a_4}}{\tilde{a}_2 + \dfrac{k'^2 a_4 - a_1}{k^2}} = A_1 \cdot \frac{\tilde{a}_2 + A_2}{\tilde{a}_2 + A_3}, \tag{19}$$

where the last equality follows from (14). Then

$$k^2 = \frac{(a_4 - a_1)(\tilde{a}_3 - \tilde{a}_2)}{(a_4 - \tilde{a}_2)(\tilde{a}_3 - a_1)} \tag{20}$$

and

$$\mathrm{sn}^2(\tilde{\varrho}) = \frac{a_4 - \tilde{a}_2}{a_4 - a_1} \tag{21}$$

hold. Note that the modulus k for the tuples (a_1, a_2, a_3, a_4) and $(a_1, \tilde{a}_2, \tilde{a}_3, a_4)$ is the same. By (17), (20), (21) and Theorem 3, $(a_1, \tilde{a}_2, \tilde{a}_3, a_4)$ is a $\mathbf{T}_n$-tuple. By (12) and (18),

$$\begin{aligned}
|a_2 - \tilde{a}_2| &= |a_4 - a_1| \cdot |\,\mathrm{sn}^2(\varrho) - \mathrm{sn}^2(\tilde{\varrho})| \\
&= |a_4 - a_1| \cdot |\,\mathrm{sn}(\varrho) - \mathrm{sn}(\tilde{\varrho})| \cdot |\,\mathrm{sn}(\varrho) + \mathrm{sn}(\tilde{\varrho})| \\
&= |a_4 - a_1| \cdot |\,\mathrm{sn}(\varrho) + \mathrm{sn}(\tilde{\varrho})| \cdot \frac{|2\,\mathrm{sn}(\frac{1}{2}\varrho - \frac{1}{2}\tilde{\varrho})\,\mathrm{cn}(\frac{1}{2}\varrho + \frac{1}{2}\tilde{\varrho})\,\mathrm{dn}(\frac{1}{2}\varrho + \frac{1}{2}\tilde{\varrho})|}{|1 - k^2\,\mathrm{sn}^2(\frac{1}{2}\varrho + \frac{1}{2}\tilde{\varrho})\,\mathrm{sn}^2(\frac{1}{2}\varrho - \frac{1}{2}\tilde{\varrho})|},
\end{aligned}$$

where in the last equation the well-known formula for $\mathrm{sn}(u) - \mathrm{sn}(v)$ is used; see, e.g., [3, (123.06)]. By (11), (15), and (17),

$$\begin{aligned}
|\varrho - \tilde{\varrho}| &= |(\lambda - \tfrac{m}{n})K + \mathrm{i}(\lambda' - \tfrac{m'}{n})K'| \\
&\leq |\lambda - \tfrac{m}{n}| \cdot |K| + |\lambda' - \tfrac{m'}{n}| \cdot |K'| \\
&\leq \frac{|K| + |K'|}{n}.
\end{aligned} \tag{22}$$

If

$$n \geq \tfrac{2}{\pi}(|K| + |K'|) =: n_1, \tag{23}$$

then, by (22),

$$|\tfrac{1}{2}\varrho - \tfrac{1}{2}\tilde{\varrho}| \leq \frac{\pi}{4}. \tag{24}$$

Thus, using Lemma 1 and (22),

$$|\,\mathrm{sn}(\tfrac{1}{2}\varrho - \tfrac{1}{2}\tilde{\varrho})| \leq \tfrac{4}{\pi}|\tfrac{1}{2}\varrho - \tfrac{1}{2}\tilde{\varrho}| \leq \tfrac{2}{\pi}(|K| + |K'|)\tfrac{1}{n}.$$

Summing up, for $n \geq n_1$, we have the inequality

$$|a_2 - \tilde{a}_2| \leq \frac{2}{n\pi}(|K| + |K'|)|a_4 - a_1|B,$$

where

$$B := |\operatorname{sn}(\varrho) + \operatorname{sn}(\tilde{\varrho})| \cdot \frac{|2\operatorname{cn}(\frac{1}{2}\varrho + \frac{1}{2}\tilde{\varrho})\operatorname{dn}(\frac{1}{2}\varrho + \frac{1}{2}\tilde{\varrho})|}{|1 - k^2\operatorname{sn}^2(\frac{1}{2}\varrho + \frac{1}{2}\tilde{\varrho})\operatorname{sn}^2(\frac{1}{2}\varrho - \frac{1}{2}\tilde{\varrho})|}. \tag{25}$$

Since $\varrho \notin \{0, K, \pm\mathrm{i}K', K \pm \mathrm{i}K'\}$, there exists an $n_2 \in \mathbb{N}$ such that $0, K, \pm\mathrm{i}K', K \pm \mathrm{i}K' \notin \mathcal{P}(n_2)$, where

$$\mathcal{P}(n_2) := \left\{ u \in \mathbb{C} : u = \mu K + \mathrm{i}\mu' K',\ |\mu - \lambda| \leq \frac{1}{n_2},\ |\mu' - \lambda'| \leq \frac{1}{n_2},\ \mu, \mu' \in \mathbb{R} \right\}.$$

Thus, the maxima

$$s^* := \max_{u \in \mathcal{P}(n_2)} |\operatorname{sn}(u)|, \qquad c^* := \max_{u \in \mathcal{P}(n_2)} |\operatorname{cn}(u)|, \qquad d^* := \max_{u \in \mathcal{P}(n_2)} |\operatorname{dn}(u)|$$

exist. By construction of $\mathcal{P}(n_2)$, for $n \geq n_2$, we have $\varrho, \tilde{\varrho}, \frac{1}{2}(\varrho + \tilde{\varrho}) \in \mathcal{P}(n_2)$. Further, we have

$$\left|k^2\right| \cdot \left|\operatorname{sn}^2(\tfrac{1}{2}\varrho + \tfrac{1}{2}\tilde{\varrho})\right| \cdot \left|\operatorname{sn}^2(\tfrac{1}{2}\varrho - \tfrac{1}{2}\tilde{\varrho})\right| \leq (s^*)^2 \cdot \frac{4}{\pi^2}(|K| + |K'|)^2 \cdot \frac{1}{n^2} \leq \frac{1}{2},$$

where the last inequality is true if

$$n \geq n_3 := \frac{2\sqrt{2}\, s^*}{\pi}(|K| + |K'|).$$

Hence, for B defined in (25),

$$B \leq \frac{4s^*c^*d^*}{\left|1 - |k^2| \cdot \left|\operatorname{sn}^2(\frac{1}{2}\varrho + \frac{1}{2}\tilde{\varrho})\right| \cdot \left|\operatorname{sn}^2(\frac{1}{2}\varrho - \frac{1}{2}\tilde{\varrho})\right|\right|} \leq 8s^*c^*d^*,$$

and altogether, for $n \geq \max\{n_1, n_2, n_3\}$, we get the inequality

$$|a_2 - \tilde{a}_2| \leq \frac{C_1}{n},$$

where

$$C_1 := \frac{16}{\pi}\left(|K| + |K'|\right)|a_4 - a_1|\, s^*c^*d^*. \tag{26}$$

Using (13) and (19), we get

$$|a_3 - \tilde{a}_3| = |A_1| \cdot \frac{|a_2 - \tilde{a}_2| \cdot |A_2 - A_3|}{|a_2 + A_3| \cdot |\tilde{a}_2 + A_3|}$$

and

$$|\tilde{a}_2 + A_3| = |a_4 - a_1| \cdot |\operatorname{sn}^2(\tilde{\varrho}) - 1/k^2|.$$

Since $u = K + \mathrm{i}K' \in \mathcal{P}$ is the only point in $\mathcal{P}$ for which $\operatorname{sn}^2(u) = 1/k^2$, by the construction of $\mathcal{P}(n_2)$,

$$s^{**} := \min_{u \in \mathcal{P}(n_2)} |\operatorname{sn}^2(u) - 1/k^2| > 0$$

holds. Thus

$$|a_3 - \tilde{a}_3| \leq \frac{C_2}{n},$$

where

$$C_2 := \frac{C_1 \cdot |A_1| \cdot |A_2 - A_3|}{|a_2 + A_3| \cdot |a_4 - a_1| \cdot s^{**}} \tag{27}$$

and A_1, A_2, A_3 are defined in (14). □

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